Algebraic roots of Newtonian mechanics: correlated dynamics of particles on a unique worldline

Vladimir V. Kassandrov and Ildus Sh. Khasanov

Institute of Gravitation and Cosmology, Russian Peoples' Friendship University, Moscow, Russia

1. Introduction. Pure algebra as the key to physical interactions?

Origins of the experimentally observable (and extremely intricated!) structure of fundamental interactions, of their laws, intensities and scale dependence still look as an enigma. It might seem that cardinal solution to this eternal problem is hidden in the geometry of physical space-time. However, the Minkowski geometry is too "soft" and allows for a wide variety of relativistic invariant interactions, if even the gauge invariance of the scheme is required. As to various geometries of *extended* space-time, at present they seem quite indefinite by themselves and, moreover, do not predetermine in any way a distinguished structure of physical dynamics.

That is why, from time to time, one can meet articles dealing with the most profound, *elementary* notions of physics and reformulations of these on the basis of geometry, algebra, number theory etc. We are aware that such attempts had been undertaken, say, by P.A.M. Dirac, A. Eddington and J.A. Wheeler.

Particularly, one of the most beautiful and striking ideas was the Wheeler-Feynman's conjecture on "one-electron Universe". This conjecture based on the notion of **a set of particles located on a single Worldline** easily explains the property of *identity* of elementary particles of one kind, the processes of annihilation/creation of a pair of "particle-antiparticle" (in which one treats a "positron" as an "electron" running backwards in time [1]) etc.

In his Nobel lecture [2], one of the creators of QED R.P. Feynman confessed that his true goal was the establishment of correlations of an ensemble of identical (pointlike or smeared, to avoid field divergences) particles *on a single Worldline* through their along-light-cone interactions and on the base of a unique Lagrange function. Unfortunately, the "one-electron Universe" paradigm had not been fully realized; the reasons for this will be revealed below.

In fact, this paradigm gains natural development in the framework of complex algebrodynamics [3, 4, 5]. In this approach one attempts to derive both the space-time geometry and principal dynamical equations for fields and particles from the properties of an exceptional algebraic structure, a sort of *space-time algebra.* Contrary to geometries, one possesses quite definite and transparent classification of such exceptional linear algebras based on the famous theorems of G. Frobenius and A. Hurvitz. For consistency with the STR and the Minkowski geometry, most often as such strucure it had been considered the algebra **Q** of *complex quaternions* [5, 6, 7].

Specifically, in the complex extension of space-time $\mathbb{C}\mathbf{M}$ – vector space of **Q** – the dynamics, even on a single Worldline, becomes quite nontrivial. Contrary to the case of real Minkowski space-time **M**, under any position and movement of an "observer", the equation of *complex light cone* – direct generalization of the *retardation equation* in **M** – always have a constant and, generally, great number of roots. These define a correspondent number of *copies* of one and the same particle detected by the observer at their different positions on a single Worldline; in [8] these copies have been named "duplicons".

In the framework of another approach, one considers a single Worldline in *real* **M** but allows for superluminar velocities of particles (tachyons) along it. In this case, the observer also encounters an arbitrary number of copies of one and the same tachyon. Possible existence of such copies-"images" had been noticed in [9] and examined in detail in [10]. Note that, contrary to the situation with duplicons in $\mathbb{C}\mathbf{M}$, the number of such *images* is not generally constant: some two of these can appear or disappear at discrete instants so that one has a simple model of the creation/annihilation process.

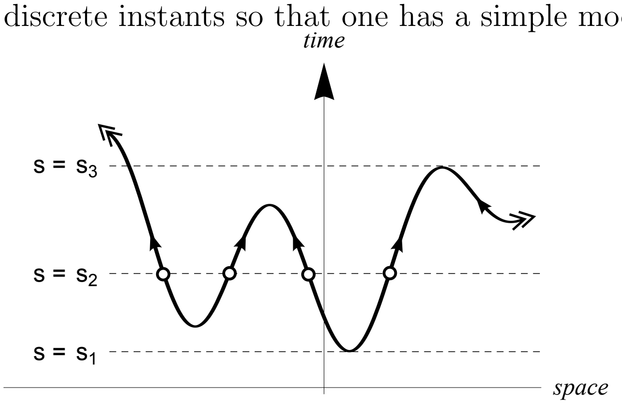


Figure 1: Generic worldline, numerous pointlike "particles" (at $s = s_2$) and creation (at $s = s_1$) or annihilation (at $s = s_3$) events

It should be noted that the Wheeler-Feynman's conjecture on the "one-electron Universe" and, especially, on "positron as a moving backwards in time electron" had been explicitly initiated by pioneer works of E.C.G. Stueckelberg [11, 12]. He assumed the existence of worldlines of general type (forbidden in the canonical STR) that contain the segments corresponding to superluminar velocities of particles' movements (figure 1). Then, corresponding (hyper)plane of equal values of the time-like coordinate $s = s_2$ intersects a worldline in a (generally, great) number of points. Physically, these form an ensemble of identical particles located on a single Worldline.

If, in the course of time, the coordinate s is assumed to increase monotonically, some two particles can appear at a particular instant $s = s_1$ (or disappear at $s = s_3$). These events model the processes of creation (annihilation) of a pair "particle-antiparticle".

Most likely, Stueckelberg [11] himself considered s as a *fourth coordinate* and does not assume it to be a (global) *evolution parameter*. As to the latter, he introduced a timelike parameter λ monotonically increasing along the trajectory and proportional to the proper time of a particle. After this, all equations of the theory can be formally represented in a relativistic invariant form. On the other hand, segments of the trajectory correspondent to opposite increments of s and λ, namely, to $ds/d\lambda < 0$

were regarded as representing the backwards-in-time motion of an antiparticle.

However, λ-parametrization is in fact parametrization for the history of *individual* particle which comes in conflict with the concept of "one-electron Universe". In order to preserve the ensemble of identical particles on a unique worldline, one should consider just s as the "true" time. Then, however, velocities of "particles" should be also measured with respect to the parameter s and are necessarily superluminar at some segments of their history (and even infinite at the annihilation points, see below). Stueckelberg himself fully comprehended this difficulty and wrote, in particular: "Ceci, et d'autres considérations d'ordre causal, nous semble être in argument important contre l'hypothése de l'existence de telles forces, malgré la covariance de leur représentation" [11, p. 592].

Subsequently, numerous approaches exploiting Stueckelberg's ideas (including his specific interpretation of the wave function, action functional and Lagrangian etc.) came to be known as *parametrized relativistic theories* (see, e.g., the review [13] and references therein). In most part of them, the additional timelike parameter had been treated as *a Lorentz invariant evolution parameter* or even as *absolute Newtonian time* [14, 15] ‡. Nonetheless, ultimate physical meaning of the variable s is still unclear. Pavsic [17] even considered it as "evolution parameter that marks an observer's subjective experience of *now*" and tried to relate this to the process of localization of a particle's wave packet (to the collapse of wave function). One way or another, multiple "particles" on a single worldline related to *one and the same value of* s, are not causally connected and cannot be *simultaneously* detected by an observer.

These and similar considerations reveal a lot of problems which arise under one's attempts to realize the "one-electron Universe" conjecture. However, the Stueckelberg-Wheeler-Feynman idea is too attractive to be abandoned at once. In account of the above mentioned Galilei-invariance of Stueckelberg's construction, at the first step it seems quite natural to consider a purely non-relativistic, three dimensional picture of processes represented at figure 1 §. **The Galilean-Newtonian picture is just that we accept in the main part of the article** and that allows for a self-consistent realization of the "one-electron Universe" conjecture.

Specifically, our main goal throughout the paper is to obtain *correlated dynamics* of identical pointlike particles from *purely algebraic properties of a single Worldline* [18] and **without any resort to the Lagrangian structure**. In this point our approach is quite different and *much more radical* that those of Stueckelberg and Wheeler-Feynman.

In the paper, instead of definition of a worldline in a habitual parametrical form (and in simplest parametrization $x_0 = s$)

$$x_a = f_a(s) \qquad (a = 1, 2, 3), \tag{1}$$

we define it (what is widely accepted for curves in mathematics) in an *implicit* form, i.e.

‡ Remarkably, in Ref. [16] invariance of Stueckelberg's action with respect to the *Galilei transformations* had been proved

§ Despite the generally accepted conviction about close connection of the annihilation/creation processes with relativistic structures through a system of three algebraic equations

$$F_a(x_1, x_2, x_3, s) = 0. \quad (2)$$

Then again, for any value of the timelike coordinate s, one generally has a whole set (N) of real roots of this system, which define a correlated kinematics $x_a = f_a^{(k)}(s)$ of the ensemble of identical pointlike singularities on a unique Worldline ‖.

It is noteworthy that the *copies* arising via this algorithm (a la Stueckelberg) *exist by themselves*. Their appearance is not related *á priori* to the existence of an "observer" or to the procedure of "registration". Thus, these identical particlelike formations do no have direct connection either with the concept of duplicons, or with the "charges-images" of Bolotovskii [10] mentioned above.

Multiple properties and "events" related to particlelike formations defined by the roots of the system (2) are considered in Section 2 and illustrated therein by a rather simple example. We restrict ourselves to *plane* motion and to *polynomial* form of *two* generating functions in (2). Particularly, we take into account not only real roots but complex conjugate roots as well: the latter turn to have independent physical sense and correspond to *another kind of particlelike formations*.

In the key Section 3 a short excursus into the methods of mathematical investigation of the solutions of system (2) (in the 2D case) of a generic polynomial type is undertaken. In the main, these methods make use of the so called *resultants* of two polynomials. After that, we demonstrate that the *Vieta's formulas* well known for a single polynomial equation, naturally arise in the 2D case too. Quite remarkably, they not only ensure the correlations between positions and dynamics of different particles in the ensemble but *reproduce in fact generic structure of Newtonian mechanics* and, in particular, lead to satisfaction of the *law of momentum conservation* (in the special inertial-like "reference frames")!

In the next Section 4, we outline some possible ways to appropriate *relativization* of the theory. In particular, we discuss the problem and possible advantages of the introduction of an external "observer" into the scheme. Alternatively, we try to define the "second time" parameter in the spirit of old conjecture of F. Klein et al. about *universal lightlike velocity of all the matter pre-elements in the extended physical space* (4D in our case). This can be treated as a reformulation of the STR and could make the structure of the principle system (2) consistent with *relativistic* mechanics.

Section 5 contains some concluding remarks on motivations and actual developments of the presented scheme. As an important part, the article contains also the Appendix. Therein, a surprisingly rich dynamics defined by a simple polynomial system presented in Section 2 is traced in detail, with the help of numerous graphical representations. *One can also see an impressve animation of the dynamics with the help of the file enclosed to the paper.*

‖ At least in simplest parametrization (1), equations for a worldline contain no trace of relativistic structure. One is thus allowed to preserve the relativistic term "worldline" in the considered Galilean-Newtonian picture

2. Two kinds of pointlike particles: algebraic kinematics

Consider for simplicity the case of *plane motion* ¶ and a curve defined implicitly through a system of two independent *polynomial* equations *with real coefficients*

$$F_1(x_1, x_2, s) = 0, \quad F_2(x_1, x_2, s) = 0, \tag{3}$$

where $s \in \mathbb{R}$ is the particular coordinate which, in addition, plays the role of *evolution parameter*: its variations will be assumed *monotonic*. As it was argued in the Introduction, one can think on s as on a Newtonian-like *absolute global time*.

We consider system (3) as the only one whose properties we shall study throughout the paper: we do not intend to supplement it by any additional equation or statement of physical or mathematical nature which does not explicitly follow from (3).

For any s, system (3) generally has a finite (N) number of roots $\{x_1^k, x_2^k\}, \quad k = 1, 2, ...N$. These define the positions of N *identical pointlike particles* at the instant s on a 2D *trajectory curve*

$$F(x_1, x_2) = 0, \tag{4}$$

whose form can be obtained from (3) after eliminaion of s and which, generally, *consists of a number of disconnected (on* $\mathbf{R}^2$*) components*. With monotonic growth of the time s, particles move along the trajectory curve with arbitrary velocities, and their number is (almost always) preserved.

However, at particular discrete instants s, say, at $s = s_0$, some two of the real roots of (3) turn into one *multiple* root and then become a pair of *complex conjugate* roots. Consequently, corresponding pair of particles merge (collide) at $s = s_0$ at some point $\{x_1^0, x_2^0\}$ and then disappear from the real slice of space. Such an "event" can serve as a model of the *annihilation process*. Conversely, at another instant some two of real roots can appear modelling the process of *pair creation*.

It should be noted nevertheless that one cannot ignore the formations which correspond to complex conjugate roots of (3) and "live" in the *complex extension* of real space. This fact will become evident in the next section while at the moment we only remark that such formations can be depicted with respect to *equal real parts* of their coordinates.

From this viewpoint, a pair of complex conjugate roots corresponds to a *composite* particle consisted of two parts coinciding on $\mathbf{R}^2$ but possessing opposite additional "tails" represented by imaginary parts of coordinates. For brevity, we shall call particlelike formations represented by real roots of the system (3) R-particles, by complex conjugate pair of roots – C-particles.

Condition for annihilation/creation events can be easily specified as that for *multiple* roots of system (3) and has the form

$$\det \|\frac{\partial F_A}{\partial x_B}\| = 0, \quad A, B, ... = 1, 2. \tag{5}$$

¶ We suspect that generalization to the physical 3D case will only be technically more complicated but none problems of principal character will arise during it

Together with (3), condition (5) defines a complete set of instants (and space locations) indicating when (and where) such events do occure.

It is now the time to present a simple example of the issues exposed above. Let us take the functions F_1, F_2 in (3), say, in the following (randomly selected) form:

$$\begin{cases} F_1(x,y,s) = -2x^3 + y^3 + sx + sy + y + 2 = 0, \\ F_2(x,y,s) = -x^3 - 2x^2y + s + 3 = 0 \end{cases} \tag{6}$$

Eliminating s, one gets the trajectory (on the real space slice) which turns to consist of *three disconnected components* (figure 2). Then via elimination of y one reduces system (6) to a single polynomial equation $P(x,s) = 0$ of the degree $N = 9$ in x and with coefficients depending on s. The latter evidently allows for full analysis and numerical calculations. As the result, one gets that, for any s, there exist 9 solutions of the system some of them being real while others – complex conjugate. Analyzing condition (5) (or, equivalently, the structure of *discriminant* of the polynomial $P(x,s)$) one concludes that there are exactly 6 "events" which correspond to the following (approximate) values of the global time s: -97.3689; -4.0246; -3; -2.7784; -2.7669; $+2932.49$. Some of these relate to annihilation (merging) events whereas others – to creations of a pair. **In the Appendix Iand in the animation file enclosed) one can find many details of the, surprisingly rich, dynamics** including processes of annihilation of two R-particles accompanied by birth of ae composite C-particle and vice versa. One observes also that the created C-"quantum" travels *between* two disconnected branches of the real trajectory, arrives at the second branch and gives there rise to a divergent pair of real R-particles (creation of a pair), see also figure 2. Remarkably, this strongly resembles the process of *exchange of quanta* specific for QFT.

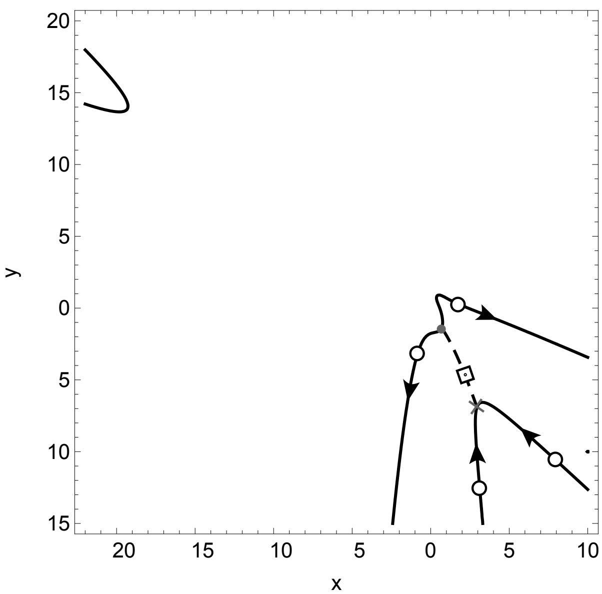


Figure 2: Three braches of the trajectory of R-particles and typical succession of events (annihilation - propagation of C-"quantum" - creation)

Two peculiar aspects of the considered algebraic dynamics can be observed. The first one is the surprisingly great "last" critical value of the time parameter $s \approx 2932.49$, despite of the numerical coefficients in (6) which all are of order 1. Thus, the "history of a Universe" defined via (6) turns to be unexpectedly long! It is not yet clear whether this property is of a particular or generic nature.

The second aspect relates to impossibility to establish a unique parametrization $x_A = x_A(\lambda), \lambda \in \mathbb{R}$ on all the three disconnected components of the trajectory. Here λ is a parameter monotonically increasing along the trajectory and exploited, in particular, by Stueckelberg himself. From this impossibility it follows that distinction of particles from antiparticles (say, "electrons" ($ds/d\lambda > 0$) from "positrons" ($ds/d\lambda < 0$)) can be established *quite independently on each branch of the trajectory*. One can speculate whether this fact could be useful in explanation of the particle/antiparticle asymmetry.

To conclude, let us obtain the expression for velocities of individual particles with respect to the global time s. Introducing canonical parametrization for an individual R-particle as $x_A = x_A(s)$ and taking then the total derivative with respect to s (denoted by a "dot") in (3) one gets

$$0 = \dot{F}_A + \frac{\partial F_A}{\partial x_B}\dot{x}_B, \tag{7}$$

whence it follows

$$\dot{x}_C = -R^A_C \dot{F}_A, \tag{8}$$

where R^A_C is the inverse matrix,

$$R^A_C \frac{\partial F_A}{\partial x_B} = \delta^B_C. \tag{9}$$

Comparing (8) with condition (5), one concludes that at the instants of annihilation/creation velocities of both particles involved in the process are necessarily infinite $^+$. In the framework of Galilean-Newtonian picture assumed throughout the paper, this property cannot cause any objection. Nonetheless, quite similar to the Stueckelberg's approach, at this point one encounters severe problems with causality and other principal statements of the STR. We consider these problems in Section 4.

3. Vieta's formulas and the law of momentum conservation

Let us concentrate first upon the procedure of resolution of the system of polynomial equations (3) of *generic type* (below we have made obvious renotations $x_1 \mapsto x,\ \ x_2 \mapsto y$),

$$\begin{cases} F_1(x,y,s) = [a_{n,0}(s)x^n + a_{n-1,1}(s)x^{n-1}y + ... + a_{0,n}(s)y^n] + ... + a_{0,0}(s) = 0, \\ F_2(x,y,s) = [b_{m,0}(s)x^m + b_{m-1,1}(s)x^{m-1}y + ... + b_{0,m}(s)y^m] + ... + b_{0,0}(s) = 0. \end{cases} \tag{10}$$

Only forms of the highest (n and m, respectively) and the least orders are written out in (10). Both polynomials are assumed to be functionally independent and irreducible, while all the coefficients $\{a_{i,j}(s), b_{i,j}(s)\}$ depend on the evolution parameter s and take values in the field of real numbers $\mathbb{R}$.

Rather surprisingly, not so much facts are known about properties of solutions of a nonlinear system of polynomial equations. Of course, some results can be taken from those for the one dimensional case. For example, it is easy to demonstrate that all the roots $\{x_0, y_0\}$ of such a system are either real (x_0 and y_0 both together) or both entering as complex conjugate pairs. However, even the problem of explicit determination of the full number of solutions of (10) over $\mathbb{C}$ from, say, the properties of coefficients and

$^+$ This can be also seen just from figure 1, since at such instants one obviously has $ds = 0, dx_A \neq 0$

degrees of the polynomials F_1, F_2 is far from being completely resolved (contrary to the one dimensional case) [19].

In practical calculations, however, it is quite possible to determine this number and evaluate approximately all the roots of (10), both real and complex conjugate. For this, the most convenient method is perhaps the method of *resultants* [19, 20]. Precisely, let $\{x_0, y_0\}$ be a solution to (10); then for $y = y_0$ being fixed both equations in (10) on x should have a *common root* $x = x_0$. Necessary and sufficient condition for this is well known:

$$R_x(y) = g_N(s)y^N + g_{N-1}(s)y^{N-1} + ... + g_0(s) = 0 \tag{11}$$

where $R[F_1(x), F_2(x), x] \equiv R_x(y)$ is the resultant of two polynomials F_1, F_2 via x taken at the condition $y = y_0$ (for simplicity the index 0 is omitted in (11) and below). Structure of the resultant (which in this case is often called *eliminant*) is represented by the determinant of *Sylvester matrix* (see, e.g., [20, 21]). Coefficients $\{g_I(s)\}$ depend on $\{a_{i,j}(s), b_{i,j}(s)\}$.

Analogously, one can exchange the coordinates, and after elimination of y arrive at the dual condition

$$R_y(x) = f_K(s)x^K + f_{K-1}(s)x^{K-1} + ... + f_0(s) = 0. \tag{12}$$

When the coefficients $a_{n,0}, a_{0,n}, b_{m,0}, b_{0,m}$ *are all nonzero*, the leading terms in (11) and (12), as a rule, * are of equal degree $K = N = mn$ (see, e.g., [22, 23, 24]). Then all their ($N = mn$) solutions over $\mathbb{C}$ can be numericallly evaluated and put in correspondence with each other to obtain N solutions $\{x_k(s), y_k(s)\}, k = 1, 2, ..N$ of the initial system (10). If some of the above 4 coefficients turn to zero, the number of solutions can be less then the maximal possible value mn. Nonetheless, in this case all the solutions still can be (approximately) obtained with the help of a computational software program Maple or Mathematica.

In order to illustrate the above presented procedure, consider the following system of equations (closely related to the previous one (6), see Section 3 below):

$$\begin{cases} F_1 = -2x^3 + y^3 + 6s^2x^2 + 3sy^2 - (6s^4 - s)x + (1 + s + 3s^2)y + \\ \qquad 2s^6 + s^2 + s + 2 = 0, \\ F_2 = -x^3 - 2x^2y + (3s^2 - 2s)x^2 + 4s^2xy - (3s^4 - 4s^3)x - 2s^4y + \\ \qquad s^6 - 2s^5 + s + 3 = 0. \end{cases} \tag{13}$$

Using the computer algebra system "Mathematica 8", we easily find the eliminant $R_x(y)$ and come to the equation

$$R_x(y) = 17y^9 + 153sy^8 + ... = 0. \tag{14}$$

Analogously, we get the dual condition

$$R_y(x) = -17x^9 + 153s^2x^8 + ... = 0. \tag{15}$$

* Precisely, if the numerical coefficient given by any of equal resultants $R[F_1^n(1, y), F_2^m(1, y), y] \equiv R[F_1^n(x, 1), F_2^m(x, 1), x]$ is *nonzero*, F_1^n and F_2^m being forms of the highest degrees (n and m, respectively), in (10)

The sets of 9 solutions of equations (14) and (15) can be now obtained and put in one-to-one correspondence to each other to give 9 solutions of the system (13). For example, at $s = 1$ the system has one real solution $\{x \approx 2.3079,\ y \approx -0.4848\}$ (defining the position of one R-particle) and 4 pairs of complex conjugate roots (corresponding to four C-particles).

We are now ready to consider **the most important issue of the present publication**, namely, the correlations of different roots and the related particles' dynamics. These correlations follow just from the *Vieta's formulas* for equations on eliminants (11),(12) ♯. The first and simplest of the Vieta's formulas (linear in roots) looks as follows:

$$\begin{cases} NX(s) := x_1(s) + x_2(s) + ...x_N(s) = -f_{N-1}(s)/f_N(s), \\ NY(s) := y_1(s) + y_2(s) + ...y_N(s) = -g_{N-1}(s)/g_N(s). \end{cases} \tag{16}$$

Obviously, quantities $\{X(s), Y(s)\}$ can be regarded as coordinates of the *center of mass* of the closed system of N identical (and, therefore, of equal masses) pointlike particles with coordinates represented by the roots $\{x_k(s), y_k(s)\}, k = 1, 2, ..N$ of the system (10) and varying in time s.

An important fact here is that complex conjugate roots also enter the l.h.p. of the condition (16) though their imaginary parts cancel and do not contribute to the center of mass coordinates. This observation makes it obvious that such roots cannot be regarded as "unphysical"; on the contrary, they should be treated as a **second type of particlelike formations** (C-particles) which "appear/disappear" in the processes of creation/annihilation of real R-particles and "move" in the space between the components of the trajectory of the latter. Only real parts of these complex conjugate roots contribute to the center of mass coordinates (and to total momentum, see below) and can be visualized in the physical space. We have exemplified such a visualization in the previous section. As to imaginary parts of such roots, they could be responsible for internal phases and corresponding frequencies of C-particles [6, 27]; however, their true meaning is vague at the present stage of consideration. Notice also that *effective mass* of a C-particle is in fact twice greater than that of an R-particle since any C-particle is represented by a *pair* of complex conjugate roots (and thus by their equal real parts on the physical space slice).

R.h.p. of equations (16) indicate that, generally, the center of mass of such closed "mechanical" system does not, generally, move uniformly and rectilinearly. However, one can treat this fact as a manifestation of *non-inertial nature* of the reference frame being choosed. Thus, one has the right to execute a *coordinate transformation* to another frame which would model the inertial properties of matter (recall that we deal only with a single "Worldline" representing "all the particles in the Universe").

In fact, it is easier to find just the distinguished reference frame in which *center of mass is at rest*. To do this, let us return back to the eliminants (11),(12) and get rid of

♯ Below we consider the generic case when the degrees of both eliminants are equal, $K = N$

the terms of the $(N-1)$-th degree, setting

$$x = \tilde{x} - (N-1) f_{N-1}(s)/f_N(s), \quad y = \tilde{y} - (N-1) g_{N-1}(s)/g_N(s). \tag{17}$$

Now one can rewrite system (10) in the new variables as

$$\tilde{F}_1(\tilde{x}, \tilde{y}, s) = 0, \quad \tilde{F}_2(\tilde{x}, \tilde{y}, s) = 0 \tag{18}$$

and consider it as describing the same closed "mechanical" system of N particles in the *center of mass reference frame*. Indeed, equations on eliminants (11),(12) in the new variables take the form

$$\tilde{R}_y(\bar{x}) = f_N(s)\tilde{x}^N + 0 + ... + \tilde{f}_0(s) = 0, \quad \tilde{R}_x(\bar{y}) = g_N(s)\tilde{y}^N + 0 + ... + \tilde{g}_0(s) = 0 \tag{19}$$

and, according to the Vieta's formulas (16), one gets

$$N\tilde{X}(s) := \tilde{x}_1(s) + \tilde{x}_2(s) + ..\tilde{x}_N(s) = 0, \quad N\tilde{Y}(s) := \tilde{y}_1(s) + \tilde{y}_2(s) + ...\tilde{y}_N(s) = 0. \tag{20}$$

Differentiating then (20) with respect to the evolution parameter s one obtains the law of conservation of the projections P_x, P_y of total momentum for a closed system of identical "interacting" particles defined by equations (18):

$$P_x := \dot{x}_1(s) + \dot{x}_2(s) + ... + \dot{x}_N(s) = 0, \quad P_y := \dot{y}_1(s) + \dot{y}_2(s) + ... + \dot{y}_N(s) = 0 \tag{21}$$

(the sign "tilde" is omitted for simplicity).

If necessary, one can now transfer to another *inertial* reference frame using a *Galilei transformation*, say, $y \mapsto y, \quad x \mapsto x - Vs, \quad V = constant$ in which the center of mass moves uniformly and rectilinearly with velocity V; specifically, one gets $X(s) = Vs, \quad Y(s) = 0$.

Repeating now the procedure of differentiation, one obtains from (21) a universal constraint on instantaneous accelerations of interacting identical particles:

$$\ddot{x}_1(s) + \ddot{x}_2(s) + ...\ddot{x}_N(s) = 0, \quad \ddot{y}_1(s) + \ddot{y}_2(s) + ...\ddot{y}_N(s) = 0, \tag{22}$$

which, for simplest case of a system of two particles, gives the third Newton's law together with definition of the forces of mutual interaction (provided the equal masses are set unit, $m_1 = m_2 = 1$):

$$f_x^{(21)} = m_1 a_x^{(1)} = \ddot{x}_1, \ f_x^{(12)} = m_2 a_x^{(2)} = \ddot{x}_2, \ f_y^{(21)} = m_1 a_y^{(1)} = \ddot{y}_1, \ f_y^{(12)} = m_2 a_y^{(2)} = \ddot{y}_2; \tag{23}$$

$$a_x^{(1)}(s) + a_x^{(2)}(s) = f_x^{(21)} + f_x^{(12)} \equiv 0, \quad a_y^{(1)}(s) + a_y^{(2)}(s) = f_y^{(21)} + f_y^{(12)} \equiv 0. \tag{24}$$

Essentially, for two particles the whole system of Newton's mechanics may be completely recovered (though the concrete form of the forces' laws themselves is not fixed by the equation of the Wordline (18)).

Consider now the case of 3 particles constituting a closed mechanical system. Then in order to resolve the universal constraint on accelerations (22) (say, along x, and analogously along y)

$$a_x^{(1)}(s) + a_x^{(2)}(s) + a_x^{(3)}(s) = 0, \tag{25}$$

one may *introduce* the forces of mutual action and reaction

$$a_x^{(1)}(s) = f_x^{(21)} + f_x^{(31)}, \quad a_x^{(2)}(s) = f_x^{(32)} + f_x^{(12)}, \quad a_x^{(3)}(s) = f_x^{(13)} + f_x^{(23)}, \tag{26}$$

which then shoud satisfy the 3-d Newton's law:

$$f_x^{(21)} + f_x^{(12)} \equiv 0, \quad f_x^{(13)} + f_x^{(31)} \equiv 0, \quad f_x^{(32)} + f_x^{(23)} \equiv 0. \tag{27}$$

However, system (26),(27) cannot be uniquely resolved with respect to the forces of mutual action-reaction. Of course, this fact is valid for any number of particles $N \geq 2$ and is of general importance. In other words, *in a closed mechanical system it is principally impossible to uniquely determine contributions of partial forces of action-reaction using only observations on accelerations of all the individual particles*! This fact (probably, not so widely known) can be regarded as the indication that, generally, the N-body problem should from the beginning be formulated at the language of *collective interactions*.

Let us now again return to consider the general construction presented above at the model of "mechanical" system consisted of $N = 9$ "particles" and defined by the equations of the Worldline (13). Since the terms of degree $8 = N - 1$ in the eliminants (14),(15) are nonzero and corresponding coefficients, moreover, depend on the time parameter s, the total momentum is not conserved so that equations (13) represent the Worldline in a non-inertial reference frame. In order to make a transition to the center of mass frame, one has to execute, according to (17), transformation of coordinates of the form

$$x = \tilde{x} + s^2, \quad y = \tilde{y} + s. \tag{28}$$

In the new variables, eliminants (14),(15) take the form

$$R_x(y) = 17y^9 + 0 + (35 + 33s)y^7 + ... = 0; \tag{29}$$

$$R_y(x) = -17x^9 + 0 + (4s - 4)x^7 + ... = 0; \tag{30}$$

whereas the defining system (13) turns to be the (already examined in the previous section) system of equations (6). It is now not difficult to check that the total momentum of all 9 particles defined by the latter is the same at every instant s and, precisely, equal to zero. Thus, equations (6) and (13) represent in fact the same ensemble of identical particles in the inertial center of mass reference frame and in a non-inertial one, respectively.

To conclude the section, it is worthy to note that besides the simplest linear Vieta's formulas (16), there exist other nonlinear ones highest of which, say, look as follows:

$$x_1(s)x_2(s)...x_N(s) = f_0(s)/f_N(s), \quad y_1(s)y_2(s)...y_N(s) = g_0(s)/g_N(s). \tag{31}$$

In principle, it is possible to find a transformation of coordinates that will do away with a number of terms in the eliminants; in this case one would have, apart from the center of mass and the related total momentum conservations, other combinations of roots (and their derivatives) which would preserve their values in time ("nonlinear integrals of motion in the framework of Newtonian mechanics"?). However, such transformations are implicit in nature (see, e.g., [21, 25]), and to find the transformed form of the defining system of equations as a whole is very difficult if possible. This problem certainly deserves further consideration.

4. Remarks on relativization of the scheme

We have demonstrated that any general system of polynomial equations like (10) completely defines a single "Wordline" and an ensemble of identical pointlike particles located on it. Their dynamics with respect to the evolution parameter s reproduces generic structure of the Newtonian mechanics and, after the choice of a special (inertial) reference frame, obeys the law of momentum conservation.

It is now necessary (especially, in account of one's claims to offer the explanation of annihilation/creation processes) to seek for possibilities of *relativization* of the theory. The formal way used for this purpose by Stueckelberg and his followers, as it was demonstrated in the Introduction, seems to be unsatisfactory since it forbids realization of the "one-electron Universe" conjecture. On the other hand, whether one regards the invariant parameter s as a "true" time (with respect to which velocities of "particles" on the Worldline should be defined), then the scheme comes into irreconcilable conflict with the principles of STR (causality problems, tachyonic behavior). Besides, the very sense of the s-parameter and its relation to other "times" (coordinate time, proper time etc.) still remains vague.

In order to remove contradictions with the STR, as the first natural step one has to explicitly introduce into the scheme an *observer* and consider the process of detection of the (R- and C-) particles. Specifically, one must supplement the system of equations like (10) (generalized to the 3D case) by the *retardation equation*

$$c^2(t-s)^2 = (x_o(t)-x)^2 + (y_o(t)-y))^2 + (z_o(t)-z))^2. \tag{32}$$

Here the functions $\{x_o(t), y_o(t), z_o(t)\}$ define the worldline of an observer while $\{x, y, z\}$ are the coordinates of the particles' Wordline implicitly depending on s via the system (10). At this step, **the fundamental constant − velocity of light c − enters the theory for the first time**. Moreover, introduction of the light cone equation (32) clarifies the meaning of s as of the *retarded time* parameter. Now, at any instant of the laboratory time t the observer receives lightlike signals from the whole set of particles located on a single Wordline but at *distinct instants of the retarded time s*. Besides, this procedure opens a possibility to escape tachyonic behavior of particles at hand. Indeed, velocities fixed by the observer with respect to his own time and to the retarded time defined by localizations of particles themselves can be quite different [26]. We remark that on a complexified space-time background, corresponding procedure was already exploited in the afore-mentioned theory dealing with the ensemble of duplicons [7, 27] and will be considered in more details elsewhere.

Another possibility to overcome superluminar velocities relates to the old conjecture of F. Klein [28], Yu.B. Rumer [29] et al. that any pre-element of matter always have in fact the same, constant in modulus velocity (equal to that of light in vacuum c) but in a *multidimensional extension of physical space*. In order to realize this idea in our scheme, one should consider the 4D Euclidean space $\mathbf{E}^4$ (with s being the fourth coordinate) and introduce the following definition of the time increment dt:

$$c^2dt^2 := c^2ds^2 + dx^2 + dy^2 + dz^2, \qquad (33)$$

which is equivalent to the statement about universal total velocity ($= c$),

$$u^2 + \vec{v}^2 = c^2, \quad (u := c\frac{ds}{dt}, \ \ \vec{v} := \frac{d\vec{r}}{dt}, \quad \vec{r} := \{x, y, z\}). \qquad (34)$$

Introduction of the Euclidean structure, instead of the habitual Minkowski geometry, looks rather marginal. However, G. Montanus [31] had demonstrated that the so called *Euclidean relativity* could reproduce the main effects of the STR. On the other hand, I.A. Urusovskii in an interesting series of papers [32, 33, 34] combined the postulate on universal total velocity (34) with the conjecture on *universal uniform rotation* of particles in the "additional" space dimensions (precisely, 3 in number in his scheme) round the circle of radius equal to their *Compton length.* These two statements have deep consequences and allow, in particular, for visual geometrical explanation of many relations of quantum theory (for this, see also [30]). As to the related group of transformations, Urusovskii demonstrated that this status can be preserved by the Lorentz group, so that his scheme had been called the "6D treatment of Special Relativity" [32].

In the framework of the scheme presented here, the Montanus-Urusovskii's approach is interesting in two aspects. The first one is rather evident: velocities of the considered particles, with respect to the newly defined time interval dt, become bounded from above and, in particular, approach maximal possible value c near the annihilation points.

The second aspect deals with relativization of the expression for momentum. From (33) it follows (as usually in the STR):

$$ds = dt\sqrt{1 - v^2/c^2}, \qquad (35)$$

so that the previous Newtonian expression for momentum (21) (with "restored" equal rest masses m)

$$P_x = m\dot{x} = m\frac{dx}{ds}, \quad P_y = m\dot{y} = m\frac{dy}{ds}, \quad P_z = m\dot{z} = m\frac{dz}{ds} \qquad (36)$$

takes now the well-known relativistic form

$$P_x = mv_x/\sqrt{1 - v^2/c^2}, \quad P_y = mv_y/\sqrt{1 - v^2/c^2}, \quad P_z = mv_z/\sqrt{1 - v^2/c^2}. \qquad (37)$$

Remarkably, the generating law of conservation of the center of mass position (20) contains no differentiations and therefore preserves its "non-relativistic" form.

We are not ready to discuss here all the consequences of introduction of the Euclidean time increment (33), the more so that some of them seem to differ from those required by the STR. It is only noteworthy that, geometrically, corresponding time interval Δt is equal to the path length (arc length of the trajectory curve) and can be calculated via explicit integration.

In account of the existence of the second kind of particles related to complex conjugate roots (C-particles), the definition of time increment (33) in fact should be generalized as follows:

$$c^2dt^2 := c^2ds^2 + (dx^2 + d\xi^2) + (dy^2 + d\eta^2) + (dz^2 + d\zeta)^2, \qquad (38)$$

where $\{d\xi, d\eta, d\zeta\}$ are the imaginary parts of increments of corresponding complex coordinates.

Finally, we note that introduction of the time increment in the form (33) makes corresponding time kinematically *irreversible*: any movement in the physical 3D or in an extended (real or complexified) space, by definition, gives rise to an increase of the time value, $dt > 0$.

5. Conclusion

The Stueckelberg-Wheeler-Feynman's conjecture about identical particles moving along a unique worldline, of course, looks attractive not only from "philosophical" viewpoint. It easily solves, say, the paradox that pointlike particles can altogether meet at some points of the physical 3D space (even for a 2D space the codimension of such an event is zero!). Moreover, the very condition that all such particles-copies belong to the same curve, turns to be a rigid restriction which requires a strongly correlated dynamics of these copies reproducing in fact the process of physical interactions.

Remarkably, after proper specification of the reference frame, any system of defining equations for the Worldline ensures (via the Vieta's formulas) correlations between the whole set of its roots which precisely correspond to the law of momentum conservation (for the closed system of two kinds (R- and C-) of particlelike formations represented by real and complex conjugate roots, respectively). This looks as an important indication to the **purely algebraic origins of the structure of (Galilean-Newtonian) mechanics and of physical interactions in general**.

Moreover, there exist some hints that structure of the forces' laws themselves can be also encoded in general properties of the unique Worldline. For instance, as far as in 1836, C.F. Gauss had made an interesting observation on the roots $\{z_k\}, \;\; k = 1, 2, ..., N$ of a single polynomial equation $F(z) = 0$ of a general form (see,e.g., [21, ch. 1]). These define a set of identical particles located at corresponding points of the $\mathbb{C}$-plane. Consider now any root z_0 of the *derivative* polynomial equation $F'(z) = 0$ (which does not coincide with a (multiple) root of the initial equation). Then it corresponds to a *libration point* (point of equilibrium) for the resultant field of *radial* forces produced by all the roots $\{z_k\}$, under the condition that *these forces be inversely proportional to the distance*, $f_k \propto 1/|z - z_k|$ (and effective "charges" of the sources are all equal). Unfortunately, we were unable to find an analogue of this remarkable property in the 3D case. However, this example indicates that even in the 2D case (and in the 3D one as well) the roots of the *derivative* equations for eliminants (11),(12), namely, $R'_x(y) = 0, \;\; R'_y(x) = 0$ define in fact a *new (third) kind of particlelike formations* whose dynamics can be correlated with others in a quite nontrivial way. We intend to consider this issue in a forthcoming publication.

Finally, it is noteworthy that any particle from the ensemble under consideration can be naturally endowed with equal (elementary) electric charge and produces an electromagnetic field of the Lienard-Wiechert type. It is especially interesting that this field undergoes an *amplification* at the points of merging (annihilation/creation) of a pair of particles, so that one has a nontrivial *caustic locus* which can be naturally regarded as a set of *quantum-like signals* perceived by an external observer [10, 7].

As to identification of the considered pointlike formations (matter pre-elements) with real particles, at present stage of investigation this, of course, seems premature. Moreover, physical particles could be detectable only at discrete instants of merging of some two or more pre-elements when only they emit a quantum-like signal. On this way one naturally comes to the concept of *dimerous electron* [7, 27] which was found to be especially useful in geometric explanation of the *quantum interference phenomena*.

Generally, at first one could make an attempt to find the reasons for "attraction" of different roots and, presumably, for their ability to form a sort of (stable) *clusters* which could really represent elementary particles, nuclei etc. At present this still looks like a hardly achievable dream though the results obtained above give an essential support to realization of the program.

Appendix

Making use of the general procedure described in Section 3, let us examine in detail the dynamics defined by the polynomial system of equations (6). The trajectory curve of particlelike formations represented by *real* roots of this system follows after elimination of the evolution parameter s, is defined by the equation

$$x^4 + 3x^3y + 2x^2y^2 - 2x^3 + y^3 + 3x + 2y - 2 = 0 \tag{A.1}$$

and consists of three disconnected components (see figure 2 in Section 2 above).

The full expressions for equations on *eliminants* $R_y(x)$ and $R_x(y)$ of the system (6) are as follows (compare with (29)):

$$\left\{ \begin{array}{l} R_y(x) = -17x^9 + (-4+4s)x^7 + (3s+25)x^6 + (4s^2+12+16s)x^4 + \\ (-3s^2-18s-27)x^3 + 27s + s^3 + 9s^2 + 27 = 0; \\ R_x(y) = 17y^9 + (35+33s)y^7 + (-6s+52)y^6 + (15s^2+34s+19)y^5 + \\ (40+8s-16s^2)y^4 + (49s+11s^2-s^3+113)y^3 + (-50s-12-18s^3-72s^2)y^2 + \\ (148s^2+28s^3+208s+48)y - 64 + s^4 - 48s^2 - 5s^3 - 96s = 0. \end{array} \right. \tag{A.2}$$

One obtains from (A.2) that at any instant s system (6) has 9 solutions some of them composing complex conjugate pairs; besides, since the terms of the 8-th degree are absent, the total momentum of the two types (R- and C-) of particles represented by real and complex conjugate roots is permanently equal to zero (the center of mass reference frame). Values of s that determine singular points for the solutions of (6) related to the annihilation/creation events correspond to *multiple roots* †† of the equations (A.2), or *common roots* of the two systems of equations

$$\left\{ \begin{array}{l} R_y(x) = 0, \\ R'_y(x) = 0, \end{array} \right. \tag{A.3}$$

††In order to determine these, one could use the explicit condition (5). We, however, prefer below another, more visual, method of discriminants

and

$$\begin{cases} R_x(y) = 0, \\ R'_x(y) = 0, \end{cases} \tag{A.4}$$

where "prime" denotes differentiation with respect to x or y, respectively.

Computing now *resultants* of the two polynomials in (A.3) or (A.4), which are in fact the so called *discriminants* of equations (A.2) one verifies that these two *have common factors*, so that critical values of parameter s are obtained from the *real roots* of the equation

$$\begin{aligned} R_{common}(s) = (s+3)^3(&-1030738720704832 - 2585288646749952s- \\ &2876632663642944s^2 - 3915728526452064s^3 - 6758379899262912s^4- \\ &7627803495311328s^5 - 5242401840993563s^6 - 2294579103345501s^7- \\ &652002779260446s^8 - 117671742918602s^9 - 12435143753367s^{10}- \\ &617360791689s^{11} - 4976985600s^{12} + 1769472s^{13}) = 0. \end{aligned} \tag{A.5}$$

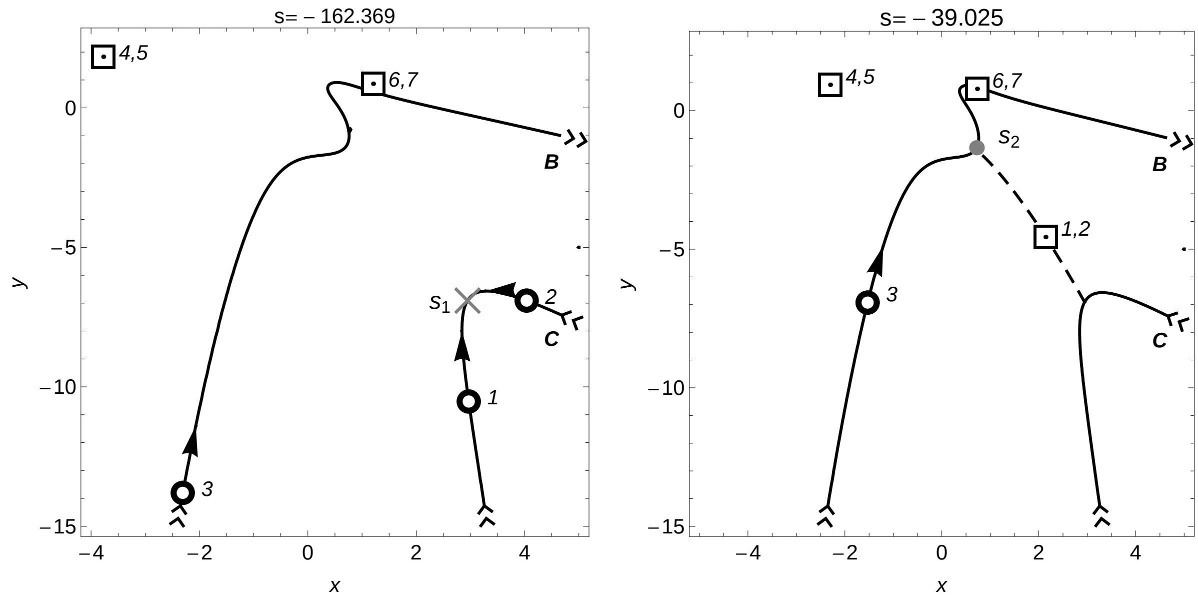
Equation (A.5) has obvious root $s = -3$ of multiplicity 3 and 13 other roots of which only 5 turn to be real. Thus, system (A.2) defines 6 critical values of parameter s at which some mergings of roots and related particlelike formations take place. Approximate critical values of s had been written out in the text (Section 2) and will be reproduced below. Corresponding coordinates of the points of merging are then readily obtained from the eliminants' equations (A.2).

Consider now graphical representation of the successive dynamics of roots of the system (6) at different values of the time parameter s. To begin with, let us agree about the notations on figures. Circles designate the positions of real roots (particles of the type R), squares – *real parts* of complex conjugate roots (particles of the type C) which are assumed thus to be located both in one and the same space point. Arrows designate the direction of motion of roots under positive increment of the parameter s. The roots are numbered in order to follow their successive dynamics and transmutations. By grey cross or circle with corresponding inscriptions $s_k,\quad k = 1, 2, ..$ one denotes the positions and instants of the annihilation or creation events, respectively. Finally, by dotted lines some segments of the projection of trajectories of complex conjugate roots onto the real plane are denoted, for visual representation of the dynamics of corresponding C-particles.

At figure A1a one sees that the real roots 1 and 2 move towards one another along the first branch of the trajectory C (4), up to their annihilation at $s_1 \approx -97.3689$.

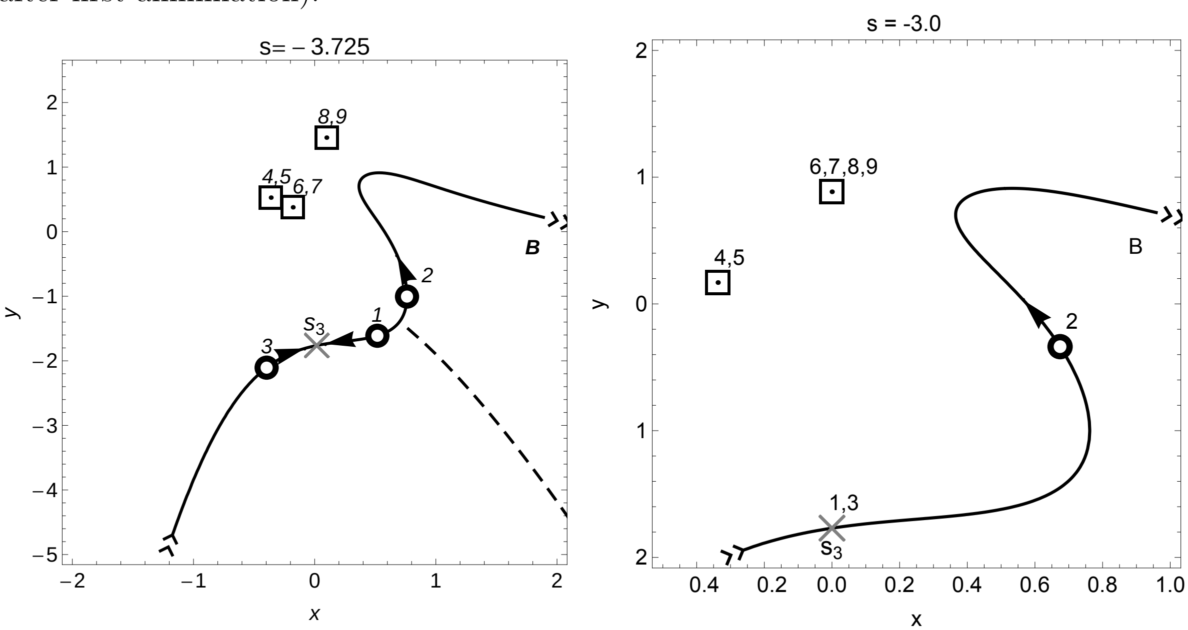
Figure A1b represents the intervening situation, when the above roots become complex conjugate and are under trasition to the other branch B when they are expected to give rise to a new pair of R-particles, at $s_2 \approx -4.025$. Note that one pair of complex conjugate roots is off the depicted space at figure A1a and figure A1b so that only 7 roots are represented therein.

At figure A2a one sees that the considered roots 1 and 2 give rise to a pair of real R-particles (1 and 2), at the branch B of the trajectory. The root 3 moves towards real root (1) and will merge with the latter at $s_3 = -3$. Note that the third pair of complex



(a) (b)

Figure A1: Disposition of roots of the system (6): (a) at $s \approx -162.37$; (b) at $s \approx -39.025$ (after first annihilation).

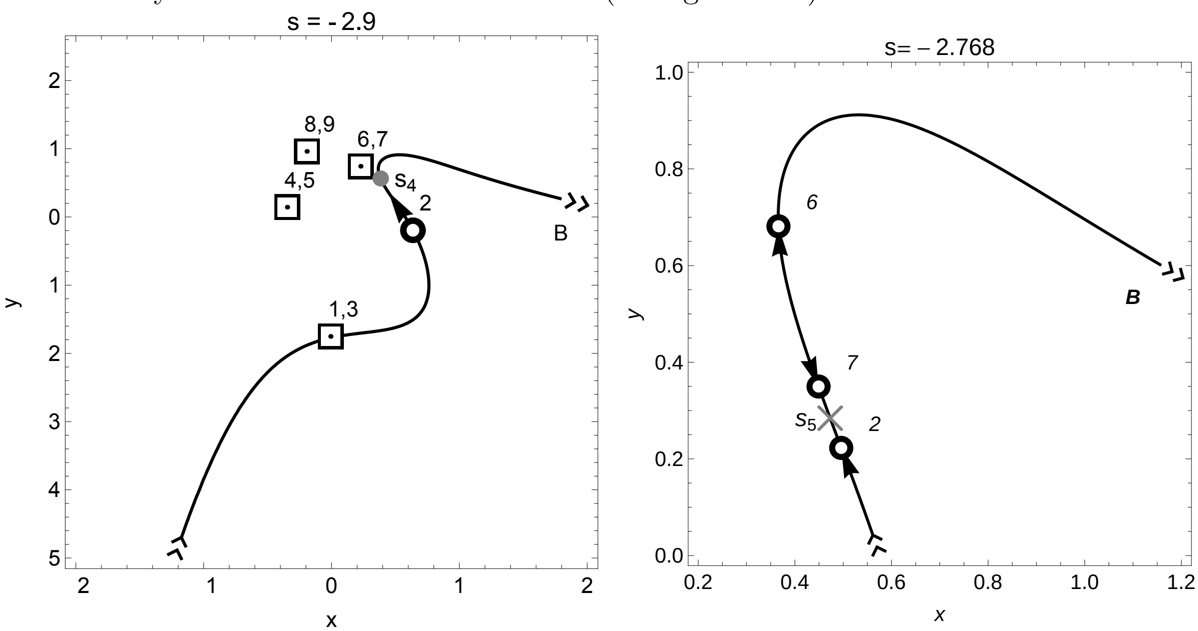


(a) (b)

Figure A2: Disposition of roots of the system (6): (a) at $s \approx -3.725$ (after first pair creation); (b) at $s = -3$ (double merging).

conjugate roots (8 and 9) appears in the space of vision so that the full number of roots (N=9) is depicted here and at the subsequent figures.

At figure A2b a peculiar situation of *double merging* is presented at $s_3 = -3$ (recall that this is the exeptional root of multiplicity 3 of the equation for "events" (A.5). At this instant, besides the annihilation of two real R-particles (1 and 3) one has the merging of two complex conjugate pairs of roots (6,7 and 8,9) which *takes place in the space exterior to the real trajectory* (i.e. in the complex extension of the "physical" 3D space). Contrary to mergings of real particles, such an event *is not accompanied by annihilation of a pair*: in what follows, the merged pairs deviate from one another, without any modification of their structure (see figure A3a).



(a) (b)

Figure A3: Disposition of roots of the system (6): (a) at $s \approx -2.9$; (b) at $s \approx -2.768$ (second pair creation).

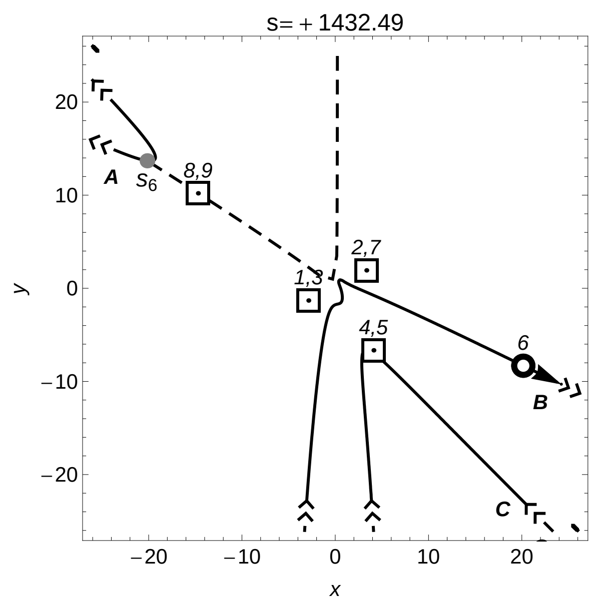


Figure A4: Disposition of roots of the system (6) at $s \approx 1432.49$ (after third annihilation).

At figure A3a one observes only one real root (2) while one pair of complex conjugate roots (6 and 7) (after divergence with the other pair (8 and 9) moves towards the branch B of the trajectory where it will give rise to a pair of real roots (6 and 7) at the next moment $s_4 \approx -2.78$.

At figure A3b the two created real particles (6 and 7) move at opposite directions along the branch B of the trajectory. At the next moment annihilation ot roots (2 and 7) at $s_5 \approx -2.77$ is expected. The pair of complex conjugate roots moves towards the third branch A of the trajectory (to be seen at the next figure) which at the moment is still "empty".

At figure A4 disposition of roots are presented in a much greater scale. After annihilation of the roots 2 and 7 only one real R-particle (6) survives on the branch B. The pair of roots 8 and 9 moves (precisely, in complex extension of space) towards the third, "empty" branch of the trajecory A where the third pair creation is expected at the future moment $s \approx 2932.49$. After this last event, there exist two real particles at branch A, one real particle at branch B and three pairs of complex conjugate roots (three C-paricles). From now on, no other merging events do exist: the dynamics is in fact over.

Full animation of the above presented dynamics is accessible with the help of the enclosed file ???

Appendix B. Acknowledgement

The authors are indebted to V.I. Zharikov for valuable comments and thorough editing of the text.

References

[1] Feynman R P 1949 *Phys. Rev.* **76** 749
[2] Feynman R P 1966 *Phys. Today* **19(8)** 31
[3] Kassandrov V V 1992 *Algebraic strusture of space-time and algebrodynamics* (Moscow: Peoples' Friend. Univ. Press) (in Russian)
[4] Kassandrov V V 1995 *Grav. & Cosm.* **1** 216; gr-qc/0007027
[5] Kassandrov V V 2007 *Space-time structure. Algebra and geometry* (Moscow: Lilia Print) p 441; arxiv: 0710:2895
[6] Kassandrov V V 2005 *Grav. & Cosm.* **11** 354; gr-qc/0602088
[7] Kassandrov V V 2009 *Phys. Atom. Nuclei* **18** 813; arxiv: 0907:5425
[8] Kassandrov V V 2005 *Proc. Int. Sci. Meet. PIRT-2005* (Moscow, Liverpool, Sunderland: Bauman Tech. Univ. Press) p 42; gr-qc/0602064
[9] Bolotovskii B M 1989 *Theoretical physics and astrophysics* (Moscow: Nauka) p 24 (in Russian)
[10] Bolotovskii B M 1990 *Sov. Phys. Usp.* **33(6)** 477
[11] Stueckelberg E C G 1941 *Helv. Phys. Acta* **14** 588
[12] Stueckelberg E C G 1942 *Helv. Phys. Acta* **15** 23
[13] Fanchi J R 1993 *Found. Phys.* **23** 487
[14] Enatsu H 1963 *Prog. Theor. Phys.* **30** 236
[15] Hotwitz L P and Piron C 1973 *Helv. Phys. Acta* **46** 316
[16] Cawley R G 1973 *Nuovo Cimento* **B16** 173
[17] Pavsic M 1991 *Found. Phys.* **21** 1005
[18] Kassandrov V V and Khasanov I Sh 2012 *XLVIII All-Russia Conf. On Problems Of Theor. Phys.* (Moscow: Russian Peoples' Friend. Univ. Press) p 120 (in Russian)
[19] Gelfand I M, Kapranov M M and Zelvinsky A V 1994 *Discriminants, resultants and multidimensional determinants* (Boston: Birkhäuser)
[20] Morozov A N and Shakirov Sh 2009 New and old results in resultant theory *Preprint* arxiv: 0911.5278
[21] Prasolov V V 2004 Polinomials *Algorithms and computation in mathematics* vol 11 (Berlin, New York: Springer)
[22] Dickenstein A. and Emilis I.Z., eds. 2005 *Solving polynomial equations: foundations, algorithms and applications* (Berlin: Springer)
[23] Brill A 1925 Vorlesungen über ebene algebraischen Kurven und algebraische Funktionen (Braunschweig: Vieweg)

[24] Kalinina E A and Utyashev A Yu 2002 *Exclusion theory* (Saint-Petersburg: St.-Petersb. Univ. Press) (in Russian)
[25] Chebotarev G N 1954 *Kazan. Gos. Univ. Uch. Zap.* **114(2)** 189 (in Russian)
[26] Rees M J 1967 *Mon. Notic. Roy. Astron. Soc.* **135** 345
[27] Kassandrov V V 2009 *Proc. Int. Sci. Meet. PIRT–2009* (Moscow, Liverpool, Sunderland: Bauman Tech. Univ. Press) p 95; arxiv: 1105.3183
[28] Klein F 1891 *Ges. Abhandfungen* vol II p 601
[29] Rumer Yu B 1956 *Studies into 5-optics* (Moscow: Gos.-Tech. Press) (in Russian)
[30] Kassandrov V V 2011 *Proc. Int. Sci. Meet. PIRT–2011* (Moscow, Liverpool, Sunderland: Bauman Tech. Univ. Press) p 131; arxiv:1209.2377
[31] Montanus J M C 2001 *Found. Phys.* **31** 1357
[32] Urusovskii I A 1996 *Zarubezhnaya radioelectronika* **3** 3 (in Russian)
[33] Urusovskii I A 1999 *Zarubezhnaya radioelectronika* **6** 64 (in Russian)
[34] Urusovskii I A 2000 *Zarubezhnaya radioelectronika* **6** 66 (in Russian)