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OPTICAL SPECTROSCOPY

# Shapiro Effect in the Light Pulse Testing of Gravitational Theories

**I. A. Urusovskii***

*Andreev Institute of Acoustics, Russian Academy of Sciences, ul. Shvernika 4, Moscow, 117036 Russia*

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**Abstract**—A possibility of using the delay of light signals passing close to the Sun for testing gravitational theories is discussed. A metric rigidly binds the observables: if the time delay of the signal is known, the angle of deviation of the observed beam, its impact parameter and perihelion in the distant observer's Galilean frame of reference should be well defined. This allows the Schwarzschild and Papapetrou metrics to be tested if the delay measurement error is no larger than a few microseconds.

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**1.** Discovery of a binary star system comprising a pulsar (with the mass of two solar masses) and a white dwarf (with the mass of half the solar mass) and measurement of the delay of the light pulses (Shapiro effect) that passed comparatively close (240 000 km) to the pulsar companion [1] open up a possibility of testing the gravitational theory by also simultaneous measuring the angle of the light beam deflection by the companion. Approximate formulas for radiolocation effects were obtained in [2]. For each beam a metric rigidly binds the observables, so that if the time delay of the signal is known, the angle of deviation of the observed beam and its impact parameter are well defined. This allows predictions of the theory to be compared with the observations if the angle of deviation or the impact parameter or the distance from the center of gravitation at the periastron point from the point of view of the distant observer is also known.

For a beam passing at the edge of the solar disc, the latter distance is known: it is equal to the Sun photosphere radius (696 000 km). Therefore, it is simpler to find out which of the known metrics agrees with the observations. For simplicity, we assume that in the Earth's orbit plane at a distance half as far from the Sun as the Earth there is a moving source of periodic light pulses or radio-waves which travel at the edge of the solar disc to the observer on Earth, who measures the pulse delay time caused by a decrease in the speed of light near the Sun.

**2.** Let us consider the simplest case of a spherically symmetric gravitational field where the star rotation effect is ignored. Then the metric in the plane running through the center of gravitation can be represented as

$$ds^2 = \gamma(r)\left\{(c\,dt)^2 - \left[\frac{c}{c_{\|}(r)}dr\right]^2 - \left[\frac{c}{c_{\perp}(r)}r d\varphi\right]^2\right\}, \quad (1)$$

where $r$ and $\varphi$ are the polar coordinates in the Galilean frame of reference, which can be regarded as the distant observer's frame of reference, $t$ is the time in this frame, $c_{\|}$, $c_{\perp}$, and $c$ are the speeds of light in the radial and perpendicular directions, and infinitely far from the center of gravitation, respectively. Considering that $ds = 0$ for the photon, we obtain

$$c\,dt = \sqrt{\left[\frac{c}{c_{\|}(r)}dr\right]^2 + \left[\frac{c}{c_{\perp}(r)}r d\varphi\right]^2}. \quad (2)$$

The Fermat principle requires the extremum of the functional

$$I = c\int_{\sigma_1}^{\sigma_2} \sqrt{\left[\frac{\sin\beta}{c_{\perp}(r)}\right]^2 + \left[\frac{\cos\beta}{c_{\|}(r)}\right]^2}\, d\sigma \quad (3)$$

between the fixed points $\sigma_1$ and $\sigma_2$ of the photon trajectory, where $d\sigma$ is the element of the trajectory arc, $\beta$ is the angle between the radial direction and the tangent to the trajectory, $d\sigma = r\,d\varphi/\sin\beta$. Expression (3) can be represented as

$$I = \int_{\varphi_1}^{\varphi_2} F(r, r')\, d\varphi,$$

*E-mail: urusovskii_ia@mail.ru

where

$$F(r,r') = \sqrt{\left[\frac{cr}{c_\perp(r)}\right]^2 + \left[\frac{cr'}{c_\parallel(r)}\right]^2}, \quad r' = \frac{dr}{d\varphi}, \quad \cot\beta = \frac{r'}{r}.$$

For the solution of the Euler equation

$$F(r,r') - r'\frac{\partial}{\partial r'}F(r,r') = \text{const},$$

which provides the extremum of this functional [3], we obtain

$$F(r,r')\left[\frac{c_\perp(r)}{cr}\right]^2 = \text{const} = \left[\frac{c_\perp(r)}{c}\right]^2 \frac{1}{r}\sqrt{\left[\frac{c}{c_\perp(r)}\right]^2 + \left[\frac{cr'}{c_\parallel(r)r}\right]^2} = \frac{c_\perp(r_\text{p})}{cr_\text{p}} = \frac{1}{D}. \tag{4}$$

Here $r_\text{p}$ is the distance from the center of gravitation to the periastron point, and $D$ is the impact parameter of the light beam; all distances are referred to the distant observer's frame of reference. From (4) there follows the relation

$$\frac{d\varphi}{dr} = \frac{c_\perp(r)}{rc_\parallel(r)}\left\{\left[\frac{cr}{c_\perp(r)D}\right]^2 - 1\right\}^{-1/2} = \frac{c_\perp(r)}{rc_\parallel(r)}\left\{\left[\frac{rc_\perp(r_\text{p})}{r_\text{p}c_\perp(r)}\right]^2 - 1\right\}^{-1/2}. \tag{5}$$

Substitution of (5) into (2) results in

$$c\,dt = \frac{c}{c_\parallel(r)}\left\{1 - \left[\frac{r_\text{p}c_\perp(r)}{rc_\perp(r_\text{p})}\right]^2\right\}^{-1/2} dr. \tag{6}$$

Integration of (5) and (6) between $r_\text{p}$ and $r$ gives the deviation of the beam from the initial direction and the signal delay

$$\begin{aligned}
\delta\varphi(r_\text{p},r) &= \int_{r_\text{p}}^{r}\left[\frac{c_\perp(y)}{c_\parallel(y)}\left\{\left[\frac{yc_\perp(r_\text{p})}{r_\text{p}c_\perp(y)}\right]^2 - 1\right\}^{-1/2} - \left[\left(\frac{y}{r_\text{p}}\right)^2 - 1\right]^{-1/2}\right]\frac{1}{y}dy,\\
\delta t(r_\text{p},r) &= \frac{1}{c}\int_{r_\text{p}}^{r}\left[\frac{c}{c_\parallel(y)}\left\{1 - \left[\frac{r_\text{p}c_\perp(y)}{yc_\perp(r_\text{p})}\right]^2\right\}^{-1/2} - \left[1 - \left(\frac{r_\text{p}}{y}\right)^2\right]^{-1/2}\right]dy.
\end{aligned} \tag{7}$$

The time for which the signal travels between the trajectory points with the coordinates $r$ and $r_\text{p}$ is

$$t(r_\text{p},r) = \frac{1}{c}\sqrt{r^2 - r_\text{p}^2} + \delta t(r_\text{p},r).$$

**3.** For the Papapetrou metric [4] (it is to this metric that the six-dimensional treatment of gravitation [4] leads under quite different assumptions) in the spherically symmetric case, we have $\gamma(r) = \exp(-r_\text{g}/r)$ and $c_\parallel(r)/c = c_\perp(r)/c = \gamma(r)$. Here the distances $r_\text{p}$ and $D$ are related by the formula $D = r_\text{p}\exp(r_\text{g}/r_\text{p})$, from which it follows that

$$r_\text{p} = D - r_\text{g}\left[1 + \frac{1}{2}\frac{r_\text{g}}{D} + \frac{2}{3}\left(\frac{r_\text{g}}{D}\right)^2 + \dots\right]. \tag{8}$$

Here $r_\text{g} = 2GM/c^2$ is the gravitational radius of the Sun, $G$ is the gravitation constant, and $M$ is the mass of the Sun. Expressions (7) take the form

$$\delta\varphi(r_{\rm p},r)=\delta\phi_{\rm P}(r_{\rm p},r)=\int\limits_u^1\left[\left\{\exp\left[2\frac{r_{\rm g}}{r_{\rm p}}(z-1)\right]-z^2\right\}^{-1/2}-\frac{1}{\sqrt{1-z^2}}\right]dz,$$
$$\delta t(r_{\rm p},r)=\delta t_{\rm P}(r_{\rm p},r)=\frac{r_{\rm p}}{c}\int\limits_u^1\left\{\exp\left(\frac{r_{\rm g}}{r_{\rm p}}z\right)\left[1-\frac{z^2}{1-z^2}\left\{\exp\left[2\frac{r_{\rm g}}{r_{\rm p}}(1-z)\right]-1\right\}\right]^{-1/2}-1\right\}\frac{dz}{z^2\sqrt{1-z^2}}, \tag{9}$$

where $u=r_{\rm p}/r$. Expanding the integrands in a series in the parameter $r_{\rm g}/r_{\rm p}$ and performing term-by-term integration, we obtain

$$\delta\varphi_{\rm P}(r_{\rm p},r)=\frac{r_{\rm g}}{r_{\rm p}}\sqrt{\frac{1-u}{1+u}}+\left(\frac{r_{\rm g}}{r_{\rm p}}\right)^2\left\{\frac{\pi}{2}-\arcsin u+\frac{1}{2}\left(\frac{1}{1+u}-3\right)\sqrt{\frac{1-u}{1+u}}+\ldots\right\},$$
$$\delta t_{\rm P}(r_{\rm p},r)=\frac{r_{\rm g}}{c}\left[\ln\left(\frac{1+\sqrt{1-u^2}}{u}\right)+\sqrt{\frac{1-u}{1+u}}\right]+\frac{r_{\rm g}}{c}\frac{r_{\rm g}}{r_{\rm p}}\left[\pi-2\arcsin u+\frac{1}{2}\left(\frac{1}{1+u}-3\right)\sqrt{\frac{1-u}{1+u}}\right]+\ldots \tag{10}$$

Papapetrou proceeds from the assumption that in the gravitational theory the simplest metric is described by only one dimensionless gravitational potential and that the product of the coefficients of the time and space parts of the metric is unity while isotropy of the speed of light is conserved. In the six-dimensional treatment of gravitation [5] this metric is obtained by applying the Fermat principle to the trajectories of elementary particles moving at the speed of light in the Compton neighborhood of the three-dimensional subspace of the six-dimensional Euclidean space.

**4.** For the Schwarzschild metric in the standard coordinates we have [2, 6] $\gamma(r)=1-(r_{\rm g}/r)$, $c_\parallel(r)/c=\gamma(r)\,c_\perp(r)/c=\sqrt{\gamma(r)}$,

$$D=r_{\rm p}\left(1-\frac{r_{\rm g}}{r_{\rm p}}\right)^{-1/2},\quad r_{\rm p}=D-\frac{r_{\rm g}}{2}\left[1+\frac{3}{4}\frac{r_{\rm g}}{D}+\left(\frac{r_{\rm g}}{D}\right)^2+\ldots\right]. \tag{11}$$

Expressions (7) take the form

$$\delta\varphi(r_{\rm p},r)=\delta\phi_{\rm st}(r_{\rm p},r)=\int\limits_u^1\left\{\left[1-z^2-\frac{r_{\rm g}}{r_{\rm p}}(1-z^3)\right]^{-1/2}-\frac{1}{\sqrt{1-z^2}}\right\}dz$$
$$=\frac{r_{\rm g}}{r_{\rm p}}\left(1+\frac{u}{2}\right)\sqrt{\frac{1-u}{1+u}}+\frac{1}{8}\left(\frac{r_{\rm g}}{r_{\rm p}}\right)^2\left\{\frac{15}{2}\left[\frac{\pi}{2}-\arcsin u\right]+\left(\frac{1}{1+u}-5\right)\sqrt{\frac{1-u}{1+u}}+\frac{3u}{2}\sqrt{1-u^2}+\ldots\right\},$$
$$\delta t(r_{\rm p},r)=\delta t_{\rm st}(r_{\rm p},r)=\frac{r_{\rm p}}{c}\int\limits_u^1\left\{\left(1-\frac{r_{\rm g}}{r_{\rm p}}z\right)^{-1}\left[1-\frac{1-(r_{\rm g}/r_{\rm p})z}{1-(r_{\rm g}/r_{\rm p})}z^2\right]^{-1/2}-\frac{1}{\sqrt{1-z^2}}\right\}\frac{1}{z^2}dz$$
$$=\frac{r_{\rm g}}{c}\left\{\ln\frac{1+\sqrt{1-u^2}}{u}+\frac{1}{2}\sqrt{\frac{1-u}{1+u}}+\frac{r_{\rm g}}{8r_{\rm p}}\left[\left(\frac{1}{1+u}-5\right)\sqrt{\frac{1-u}{1+u}}+15\left(\frac{\pi}{2}-\arcsin u\right)\right]+\ldots\right\}. \tag{12}$$

For the Schwarzschild metric in the isotropic coordinates we have [2, 6]

$$\gamma(r)=\left[\frac{4-(r_{\rm g}/r)}{4+(r_{\rm g}/r)}\right]^2,\quad \frac{c_\parallel(r)}{c}=\frac{c_\perp(r)}{c}=\left(1-\frac{r_{\rm g}}{4r}\right)\left(1+\frac{r_{\rm g}}{4r}\right)^{-3},$$
$$D=r_{\rm p}\left(1+\frac{r_{\rm g}}{4r_{\rm p}}\right)^3\left(1-\frac{r_{\rm g}}{4r_{\rm p}}\right)^{-1},\quad r_{\rm p}=D-r_{\rm g}\left[1+\frac{7}{16}\frac{r_{\rm g}}{D}+\frac{9}{16}\left(\frac{r_{\rm g}}{D}\right)^2+\ldots\right], \tag{13}$$

and expressions (7) take the form

$$\delta\varphi(r_{\rm p},r)=\delta\varphi_{\rm iso}(r_{\rm p},r)=\int\limits_u^1\left[\left\{\left[\frac{4+(r_{\rm g}/r_{\rm p})z}{4+(r_{\rm g}/r_{\rm p})}\right]^6\left[\frac{4-(r_{\rm g}/r_{\rm p}}{4-(r_{\rm g}/r_{\rm p})z}\right]^2-z^2\right\}^{-1/2}-\frac{1}{\sqrt{1-z^2}}\right]dz$$
$$=\frac{r_{\rm g}}{r_{\rm p}}\sqrt{\frac{1-u}{1+u}}+\frac{1}{2}\left(\frac{r_{\rm g}}{r_{\rm p}}\right)^2\left\{\frac{3}{16}\left[\frac{\pi}{2}-\arcsin u\right]+\left(\frac{1}{1+u}+\frac{3}{8}\right)\sqrt{\frac{1-u}{1+u}}+...\right\},$$
$$\delta t(r_{\rm p},r)=\delta t_{\rm iso}(r_{\rm p},r)=\frac{r_{\rm p}}{c}\int\limits_u^1\left\{\frac{(1+r_{\rm g}z/4r_{\rm p})^3}{1-r_{\rm g}z/4r_{\rm p}}\left[1-\left(\frac{1+r_{\rm g}/4r_{\rm p}}{1+r_{\rm g}z/4r_{\rm p}}\right)^6\left(\frac{1-r_{\rm g}z/4r_{\rm p}}{1-r_{\rm g}/4r_{\rm p}}\right)^2z^2\right]^{-1/2}-\frac{1}{\sqrt{1-z^2}}\right\}\frac{1}{z^2}\,dz$$
$$=\frac{r_{\rm g}}{c}\left[\ln\frac{1+\sqrt{1-u^2}}{u}+\sqrt{\frac{1-u}{1+u}}+\frac{r_{\rm g}}{2r_{\rm p}}\left\{\frac{31}{16}\left[\frac{\pi}{2}-\arcsin u\right]-\left(\frac{9}{4}+\frac{u}{1+u}\right)\sqrt{\frac{1-u}{1+u}}\right\}+...\right]. \quad (14)$$

**5.** For the Papapetrou metrics the total beam deviation and the signal delay are

$$\Delta\varphi_{\rm P}=\delta\varphi_{\rm P}(r_{\rm p},r_{\rm s})+\delta\varphi_{\rm P}(r_{\rm p},r_{\rm o}),$$
$$\Delta t_{\rm P}=\delta t_{\rm P}(r_{\rm p},r_{\rm s})+\delta t_{\rm P}(r_{\rm p},r_{\rm o}), \quad (15)$$

where $r_{\rm s}$ and $r_{\rm o}$ are the distances from the center of gravitation to the source point and the observation point, respectively. For the other two metrics considered above the corresponding formulas are obtained by replacing the subscript P in (15) with the subscripts st or iso. The numerical values of the shifts for the beam moving at the edge of the solar disc are

$$\Delta\varphi_{\rm P}=1.738'',$$
$$\Delta\varphi_{\rm st}=1.744'',$$
$$\Delta\varphi_{\rm iso}=1.738'',$$
$$\Delta\varphi_{\rm st}-\Delta\varphi_{\rm P}=0.006'',$$
$$\Delta\varphi_{\rm iso}-\Delta\varphi_{\rm P}=1.9''{\times}10^{-6},$$
$$\Delta t_{\rm P}=132.2\,\mu{\rm s},$$
$$\Delta t_{\rm st}=122.4\,\mu{\rm s},$$
$$\Delta t_{\rm iso}=132.2\,\mu{\rm s},$$
$$\Delta t_{\rm P}-\Delta t_{\rm st}=9.8\,\mu{\rm s},$$
$$\Delta t_{\rm P}-\Delta t_{\rm iso}=1.2{\times}10^{-5}\mu{\rm s}.$$

It is seen that in the Schwarzschild metric in the standard coordinates the signal delay under the conditions in question is almost 10 $\mu$s smaller than in the compared cases. With the measurement error no larger than a few microseconds, it is possible to observe this difference in signal delay, which allows preference to be made in favor of either the Schwarzschild metric in the standard coordinates or any of the other two metrics.

However, for comparison of the metric gravitational theories with the observations, the coordinates in these theories should be expressed in terms of measurable quantities. One of these quantities is the metric coefficient $\gamma(r)$ characterizing time dilation in the neighborhood of a massive body and gravitational displacement of the source radiation frequency. For the Papapetrou metric, $r_{\rm g}/r=-\ln\gamma(r)$ and $r_{\rm g}/r_{\rm p}=-\ln\gamma(r_{\rm p})$; for the Schwarzschild metric in the standard coordinates, $r_{\rm g}/r=1-\gamma(r)$ and $r_{\rm g}/r_{\rm p}=1-\gamma(r_{\rm p})$; and for the Schwarzschild metric in the isotropic coordinates,

$$\frac{r_{\rm g}}{r}=4\,\frac{1-\sqrt{\gamma(r)}}{1+\sqrt{\gamma(r)}},$$
$$\frac{r_{\rm g}}{r_{\rm p}}=4\,\frac{1-\sqrt{\gamma(r_{\rm p})}}{1+\sqrt{\gamma(r_{\rm p})}}.$$

If the coefficients $\gamma(r_{\rm p})$ for all metrics under consideration are set equal to this coefficient in the Papapetrou metric, $\Delta t_{\rm st}$ will decrease only by $2.4{\times}10^{-9}\,\mu$s and $\Delta t_{\rm iso}$ by $3.3{\times}10^{-6}\,\mu$s. Variation in small values $r_{\rm g}/r$ at the source and reception points is all the more negligible because variation in the radial coordinate by a value on the order of $r_{\rm g}$ changes $r_{\rm g}/r$ only by a value on the order of $(r_{\rm g}/r)^2$.

For the local observer, the gravitational acceleration has the form [5]

$$g_{\rm loc}(r)=c_{\parallel}(r)\,\frac{c}{\gamma(r)}\frac{d\sqrt{\gamma(r)}}{dr}.$$

In particular,

$$g_{\rm loc}(r)=c^2\frac{r_{\rm g}}{2r^2}\exp\left(-\frac{r_{\rm g}}{2r}\right)$$

for the Papapetrou metric,

$$g_{\rm loc}(r)=c^2\frac{r_{\rm g}}{2r^2}\left(1-\frac{r_{\rm g}}{r}\right)^{-1/2}$$

for the Schwarzschild metric in the standard coordinates, and

$$g_{\rm loc}(r) = c^2 \frac{r_{\rm g}}{2r}\left(r - \frac{r_{\rm g}}{4}\right)^{-1}\left(1 + \frac{r_{\rm g}}{4r}\right)^{-3}$$

for the Schwarzschild metric in the isotropic coordinates. Now, if $g_{\rm loc}(r)$ for all metrics under consideration is set equal to that in the Papapetrou metric, $\Delta t_{\rm st}$ will decrease only by $2.0\times10^{-8}\,\mu$s and $\Delta t_{\rm iso}$ by $3.4\times10^{-6}\,\mu$s. It is evident that the effect produced on the signal delay time by the dependence of the above three observables on the choice of the metric can be ignored in the experiment under consideration.

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